Invitation to Feynman Integrals and Multiple Zeta Values

Lectures by Johannes M. Henn (Max Planck Institute for Physics, Munich),
Tohoku University, 1–3 December 2026.

Four lectures for an audience of mathematicians, on what Feynman integrals are, what functions and numbers they produce, and a positivity property those functions have. No physics is assumed.

Lecture 1 (1 December, 90 minutes): *Feynman integrals in two languages.*

Quantum field theory makes its predictions by computing integrals associated with graphs: the integrand is read off the graph by a combinatorial procedure, and the integral is a function of parameters, the masses and momenta of the particles. Physicists have built a large machinery for evaluating these integrals, from graph polynomials and integration by parts to differential equations, and much of it is familiar mathematics in an unfamiliar vocabulary. This first lecture builds a dictionary between the two vocabularies, entry by entry, on a single example that is simple enough to be worked out in full and is carried through all four lectures: the graph with two vertices joined by two edges, whose Feynman integral is, in parametric form,

$$I(s,m^2) \;=\; \int_{x_1,x_2\geq 0} \delta(1-x_1-x_2)\,\mathcal{U}^{2\epsilon}\,\mathcal{F}^{-1-\epsilon}\,dx_1\,dx_2\,, \qquad \mathcal{U}=x_1+x_2\,, \quad \mathcal{F}=s\,x_1x_2+m^2\,(x_1+x_2)^2.$$

Here $\mathcal{U}$ and $\mathcal{F}$ are the graph polynomials, s and m^2 are the parameters, and ϵ is the dimensional regulator. For rational s and m^2, each coefficient of the expansion of I in ϵ is a period in the sense of Kontsevich and Zagier.

Lecture 2 (2 December, 60 minutes): *From a differential equation to multiple zeta values.*

The integral I of the first lecture belongs to a family of integrals of the same type that spans a two-dimensional vector space over the rational functions in s, m^2 and ϵ, and a basis F of this space satisfies a first-order system of linear differential equations in s. A change of basis, suggested by the structure of the integrand, and a change of variable from s to x bring the system to the Fuchsian form

$$dF \;=\; \epsilon\left[A_0\,d\log x + A_1\,d\log(1-x)\right]F,$$

with constant 2×2 matrices A_0 and A_1, in which ϵ appears only as an overall factor. We solve this system order by order in the dimensional regulator ϵ (the twist exponent). The solutions are iterated integrals of the two $d\log$ forms, that is, multiple polylogarithms in one variable, and their weight equals the order in ϵ. Comparing the regularised solutions at the singular points $x=0$ and $x=1$, as in the construction of the Drinfeld associator, produces shuffle-regularised multiple zeta values. We close with an overview of the function spaces that appear in more general Feynman integrals, and of the numbers beyond multiple zeta values to which they lead.

Lecture 3 (2 December, 60 minutes): *Positivity: a new question about familiar functions.*

In the region of the kinematic variables where the graph polynomials are positive on the integration domain, a Feynman integral is the integral of a positive function. This leaves a trace in the answer: as a function of the kinematic variable it is completely monotone, and often a Stieltjes function. This implies that its expansion coefficients are, up to signs, the moments of a positive measure, and that rational approximations to it converge with two-sided bounds. The lecture establishes these properties and describes how they are being used to constrain and approximate Feynman integrals. We also discuss the question which elements of a given function space, for instance which multiple polylogarithms, are Stieltjes functions.

Lecture 4 (3 December, 60 minutes): *Positivity and Aomoto's polylogarithms.*

This lecture gives a different interpretation of where the positivity of Feynman integrals found in the third lecture can come from, and ends with an outlook towards number theory: the relation between Padé approximants of Stieltjes functions and rational approximations of zeta values, in the spirit of the work of R. Apéry and F. Brown. A positive geometry is a region carrying a canonical differential form whose singularities lie on its boundary. Integrating the form of one region over another gives Aomoto's polylogarithms, and the Feynman integral of the first lecture is the simplest case. Residues of the form give the discontinuities, derivatives give the differential equations, and two triangulations of one region give a functional equation such as the five-term relation of the dilogarithm. The positivity found in the previous lecture can be understood from that of the canonical form.

Suggested preparation, for anyone who wants to read ahead: S. Weinzierl, *Feynman Integrals*, arXiv:2201.03593, sections 2.5.2–2.5.3 and 3.1, for the parametric representation and the graph polynomials.